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Interactions of domain walls of SUSY Yang-Mills as D-branes

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Abstract: Domain walls in supersymmetric Yang-Mills are BPS configurations which preserve two supercharges of the parent theory and so their tensions are known exactly. On the other hand, they have been described as D-branes for the confining string. This leads to a description of their collective dynamics in terms of a $2+1$-dimensional gauge theory with two supersymmetries and a Chern-Simons term. We show that this open string description can capture the qualitative behaviour of the forces between the domain walls for an arbitrary configuration of n walls at leading order in $1/N$, extending earlier calculations for two walls. The potential admits a supersymmetric bound state when the n walls are all coincident and asymptotes to a constant at large separation with an n dependence which agrees perfectly with the exact tension formula.

Keywords: 1/N Expansion, Nonperturbative Effects, Supersymmetry and Duality, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Supersymmetric gauge theories are fascinating because they have all the physical properties of QCD itself in a situation where some quantities, the holomorphic ones, can be calculated exactly. For instance, this includes the condensates of lowest components of chiral operators like the gluino condensate. This leads to an exact description (modulo some caveats) of the vacuum structure of the theory. For gauge group $SU(N)$ there are N discrete vacua for which

$$\langle\lambda\lambda\rangle_j = N\Lambda^3 \exp\left(i\frac{2\pi j}{N}\right) , \tag{1.1}$$

$j = 0, 1, \ldots, N-1$. However if one is completely honest, impressive though this holomorphic data is it only scratches the surface of the gauge theory. One should strive for more. In theories with extended supersymmetry, one can make exact statements about BPS states which preserve a certain proportion of the supersymmetry, usually one half. The mass of such states is then determined by a central charge in the supersymmetry algebra. In theories with minimal supersymmetry in four dimensions, however, BPS particles states do not exist, since there is no appropriate central charge in the supersymmetry algebra. Nevertheless, there is a tensorial central charge which allows co-dimension objects to be BPS. Dvali and Shifman [1] showed that the domain walls which separate the discrete vacua are such BPS objects and the supersymmetry algebra yields an exact formula for the tension. For the wall which separates the $j^{\rm th}$ and $j+n^{\rm th}$ vacua (the label is to be understood modulo N) the tension is

$$T_n = \frac{N^2\Lambda^3}{4\pi^2}\sin\frac{\pi n}{N} . \tag{1.2}$$

$n = 1, \ldots, N-1$. It is remarkable that this formula includes all the quantum effects and is exact. It follows from this simple formula that there must be forces between domain walls. Imagine a configuration of two parallel plane domain walls with vacua j to the left, $j+n_1+n_2$ to the right, and $j+n_1$ in between. Since $T_{n_1}+T_{n_2} > T_{n_1+n_2}$ it is clearly energetically advantageous for the domain walls to move together and squeeze away the vacuum in the middle to form a bound state. Of course the above argument doesn't tell us what the force is, just that the potential must rise from zero, since the bound state is BPS, and must asymptote to a constant to account for the binding energy per unit area $\Delta T_n = nT_1 - T_n$. It must be possible to interpret the force between the walls in terms of the exchange of the particle states in the theory, in this case from the tower of glueballs. This description should be good at large distances, compared with Λ^{-1}, and the potential will have the behaviour [2]

$$V(X) \underset{X\to\infty}{=} V_0 + \sum_i C_i e^{-M_i X} \,. \tag{1.3}$$

For short distances, $X \ll |\Lambda|$ and the sum over the tower of glueballs will be less useful. *A priori*, since the domain walls are non-perturbative objects, it is not clear how to describe the short distance potential.

If we were working strictly within the confines of field theory, then this would probably be the end of the story. However, the gauge theory can be engineered in string theory and these constructions imply that the domain walls are precisely D-branes for the confining string [3] (see also [4] for a recent discussion). This means that the forces between the walls can–at least in principle—be determined by considering open string interactions between the walls. In particular, this description should be valid at small separations. The light degrees-of-freedom of n walls (the approximate moduli) are described by a $U(n)$ gauge theory on the walls with a single (real) adjoint scalar field describing the fluctuations of the walls in the transverse direction. The action describing these light fields will be some very complication Born-Infeld type theory interacting with higher mass string states. However, Acharya and Vafa suggested that the truncation of this theory to the terms most relevant at low energy was a supersymmetric Yang-Mills-Chern-Simons theory [5] :

$$\begin{aligned}\mathcal{L} = \frac{1}{2g^2}\mathrm{Tr}\Big(&- (D_i\phi)^2 - \tfrac{1}{2}(F_{ij})^2 - i\chi \not{D}\chi - i\psi \not{D}\psi - 2\lambda[\phi,\psi] \\ &+ N\big(\epsilon_{ijk}(A^i\partial^j A^k + \tfrac{1}{3}A^iA^jA^k) + i\chi\chi\big)\Big)\,.\end{aligned} \tag{1.4}$$

The theory has $\mathcal{N}=1$ supersymmetry (2 supercharges) and the field are organized in to two multiplets, (A_i, χ_α) and (ϕ, ψ_α). As we have already said, this theory will be subject to all kinds of corrections coming from stringy effects and it is far form clear whether it has any range of validity at all. In particular, in [6] we argued that the topological mass scale $m = g^2N$ set by the level of the Chern-Simons term is of order the string scale, *i.e.* the renormalization group scale Λ of the bulk $SU(N)$ theory, and so there is no actually no mass hierarchy between the fields in (1.4) and other modes of the open string or corrections coming from the truncation of the Born-Infeld action to the Yang-Mills action. However, it seems that the Acharya-Vafa theory by itself is useful at least for calculating the multiplicities of domain walls. But this is an index theory calculation and hence is probably immune to the stringy corrections.[1] Our attitude is that, baring some unexpected miracle, the Acharya-Vafa theory will not, by itself, describe the interactions between domain walls exactly, however, it is worth investigating to see whether it has the right qualitative features.

[1]Although, even here there is an interesting puzzle for the theory with gauge group $SO(N)$; see [7].

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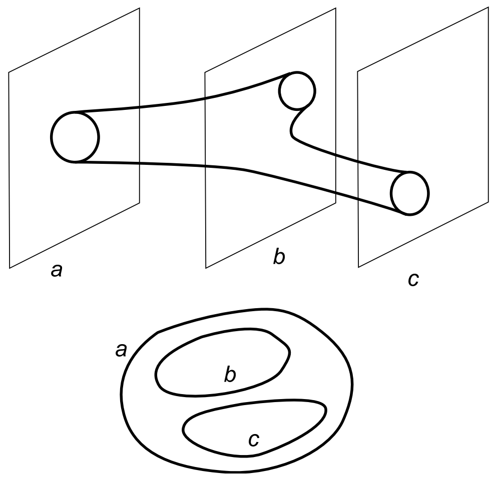

Figure 1: A thickened two-loop graph and the associated open string "pants diagram" involving three walls.

As usual, string perturbation theory corresponds to the large N expansion in the $SU(N)$ gauge theory since $g_s \sim 1/N$. The $1/N$ expansion corresponds to perturbation theory on the walls since the wall coupling constant $g^2 \sim \Lambda/N$ [6]. String perturbation theory therefore corresponds to perturbation theory of the domain wall theory. At leading order, $g^0 \sim 1/N^0$ we have the annulus diagram which vanishes because of supersymmetry. This means that the leading order interaction (in $1/N$) interaction term between n domain walls is $\sim n^3$. This is counter-intuitive since, naively, one would expect a binary interaction giving a dependence $\sim n^2$. The resolution of the puzzle is that the first non-vanishing diagram is the "pants diagram" in figure (1). In particular, it is clear that at this order, in the stringy picture, interactions involve either pairs, $a = c \neq b$, or triples, a, b and c all distinct, of walls. More generally, at order $1/N^p$ interactions can involve at most $p+2$ walls. It augurs well for an underlying string theory that this dovetails with the binding energy per unit area of n walls that follows from the exact BPS tension formula (1.2)

$$\Delta T_n = nT_1 - T_n = \frac{N^2\Lambda^3}{4\pi^2}\left(n\sin\frac{\pi}{N} - \sin\frac{\pi n}{N}\right) = \frac{\Lambda^3}{4}\sum_{p=1}^{\infty}(-1)^{p+1}\frac{n(n^{2p}-1)}{(2p+1)!}\left(\frac{\pi}{N}\right)^{2p-1}. \tag{1.5}$$

The n dependence of the term at order $1/N^{2p-1}$ is

$$\frac{n(n^{2p}-1)}{(2p+1)!} = \sum_{j=2}^{2p+1} c_j \binom{n}{j}, \tag{1.6}$$

which can be interpreted as a sum over contributions from groups of j walls where $2 \leq j \leq 2p+1$. Let us focus on the case leading $1/N$ term ($p=1$). In this case we have $\frac{n^3-n}{6} = \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2}$. The interpretation is that there are two classes of contributions from the "pants diagram" Fig. (1): one where the three strings end on different walls and one where two strings end on one all and the remaining one ends on a different wall. The exact tension formula implies that the relative weighting of these two contributions is one-to-one. It is remarkable that the calculation presented in the next section, actually confirms the above expectation: the various field theory contributions sum to a potential whose asymptotic behaviour predicts an $n^3 - n$ dependence at the order $1/N$.

Another puzzle that follows from (1.5) is why the expansion is in odd powers of $1/N$. In string theory D-branes interactions involve all powers of g_s and not only odd powers of g_s. Due to the similarity between domain walls and D-branes we expect that the interaction potential between domain walls would include odd *and even* powers of $1/N$. However, the exact tension (1.5) exhibits only odd powers of $1/N$. Logically, it is possible that the interaction *does* include even powers of $1/N$ and only the tension does not. However, this seems somewhat Machiavellian; in addition, as was already mentioned, the explicit computation shows that the zeroth order contribution to the potential vanishes identically. This is a hint that all the even power contributions vanish identically due to some symmetry. This puzzle deserves further investigation.

In the next section, we calculate the potential between n walls at order $1/N$ via a two-loop calculation in the domain wall theory. This is generalization of the calculation in [6] for the case of two walls.

2. The multi-wall potential at 2-loops

In this section, we consider the interactions between a set of n parallel domain walls with arbitrary separations X_a. From the point-of-view of the theory on the domain wall, we are at an arbitrary point on the Coulomb branch. In the quantum theory, we expect that this branch is lifted and that there will be forces between the walls. From the domain wall perspective these forces appear as a non-trivial Coleman-Weinberg effective potential on the Coulomb branch once the massive degrees-of-freedom have been integrated out.

On the Coulomb branch, the $U(n)$ gauge symmetry is broken to $U(1)^n$. The overall $U(1)$ is completely decoupled form the remaining degrees-of-freedom and so we can ignore it and work with a $SU(n)$ theory instead. After symmetry breaking,

$$\phi = \begin{pmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_n \end{pmatrix}, \qquad \varphi_a \sim \Lambda^2 X_a\,, \tag{2.1}$$

only the diagonal components of ϕ and ψ are massless, all the other fields either gain a mass through the Higgs mechanisms, or have a topological mass coming from the Chern-Simons term, or a combination of a Higgs and topological mass. We discuss them in turn below:

(1) Gauge bosons. The off-diagonal components A_i^{ab}, $a \neq b$, charged under the a pair of unbroken $U(1)$'s, have a complicated propagator which reflects a mixture between the Higgs effect and the topological mass arising from the Chern-Simons term.[2] In Euclidean

[2] A good reference for Yang-Mills-Chern-Simons theories is the review [8].

space, which we now use throughout, and Landau gauge, the propagator is

$$\Delta_{ij}^{ab}(p) = \frac{(\delta_{ij} - p_i p_j/p^2)(p^2 + \varphi_{ab}^2) - m\epsilon_{ijk}p_k}{(p^2 + m_{ab}^{(+)2})(p^2 + m_{ab}^{(-)2})}\,, \tag{2.2}$$

We have introduced the notation $\varphi_{ab} \equiv \varphi_a - \varphi_b$ and defined the masses

$$m_{ab}^{(\pm)} = \sqrt{\varphi_{ab}^2 + m^2/4} \pm m/2\,. \tag{2.3}$$

The diagonal components, neutral under the unbroken gauge group, A_i^{aa} only have a topological mass. The propagator is still given by (2.2) since $\varphi_{aa} = 0$ and $m_{aa}^{(\pm)} = m$.

(2) Scalars. The neutral components ϕ^{aa} are the massless Higgs fields while ϕ^{ab} are the would-be Goldstone Bosons and so are massless in Landau gauge.

(3) Fermions. It is convenient to amalgamate the two 2-component fermion fields which are off-diagonal in colour indices into a 4-component field:

$$\Psi^{ab} = \begin{pmatrix} \chi_\alpha^{ab} \\ \psi_\alpha^{ab} \end{pmatrix}\,. \tag{2.4}$$

In Euclidean space, the inverse propagator can then be written in 2×2 block form[3]

$$\left(\Delta_F^{ab}(p)\right)^{-1} = \begin{pmatrix} m - ip\cdot\sigma & i\varphi_{ab} \\ -i\varphi_{ab} & -ip\cdot\sigma \end{pmatrix}\,. \tag{2.5}$$

The remaining massive fields χ_α^{aa} have mass m.

In addition to these fields and their interactions, we have to add the usual gauge fixing terms and associated ghosts. The vertices are those of a conventional spontaneously broken gauge theory except that the Chern-Simons term (1.4) modifies the momentum dependence of the three gauge vertex to

$$(p_1 - p_2)_k\delta_{ij} + (p_2 - p_3)_i\delta_{jk} + (p_3 - p_1)_j\delta_{ik} - m\epsilon_{ijk} \tag{2.6}$$

in Euclidean space. Note that in Euclidean space the Chern-Simons term is pure imaginary.

The effective potential as a function of the VEVs φ_a (which become the field of the low-energy effective action) is obtained by integrating out all the massive modes: that is every field except ϕ^{aa} and ψ^{aa}. In perturbation theory, the contribution is given by summing all the vacuum graphs with massive fields propagating in the loops. It is straightforward to verify that the one-loop contribution vanishes identically due to the mass degeneracies entailed by supersymmetry. At the two loop level, there are two kinds of vacuum graph; namely, the sunset and the figure-of-eight. Each sunset graph involves three particles with charges (ab), (bc) and (ac), while a figure-of-eight involves two particles with charges (ab) and (ac). Once the diagrams are thickened out to become open string diagrams they both have the same topology and a, b and c become Chan-Paton factors associated to three domain walls as illustrated in Figure (1). Although the theory is finite, each separate

[3] Here, $\sigma_i = (\tau^1, \tau^2, \tau^3)$.

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graph is divergent and must be regularized. Since we wish to preserve supersymmetry we use the dimensional reduction regularization scheme where loop momenta propagate in d dimensions, while the tensor and spinor structure is appropriate to 3 dimensions. If the diagrams are calculated correctly, the poles in $d-3$ cancel due to supersymmetry to leave a finite result. It should also be possible to do the calculation in a real superspace formalism.

The loop integrals can all be calculated using the algorithm explained in the Appendix of [6]. The contributions from individual diagrams are in general very lengthy and since they have no real intrinsic meaning on their own we do not write down their contributions explicitly. However, once added together, enormous cancellations and simplifications occur and for this reason will only quote the end result for the effective potential:

$$V_{\text{2-loop}}(\varphi_a) = \frac{g^2 m^2}{16\pi^2} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \frac{\varphi_{ab}\varphi_{cb}}{\sqrt{(4\varphi_{ab}^2 + m^2)(4\varphi_{bc}^2 + m^2)}} . \tag{2.7}$$

It is worth emphasizing that the result depends on a sum over triplets of $U(n)$ indices since, as we have already pointed out, individual diagrams involve at most three different Chan-Paton factors.

3. Discussion

Our result for the potential can be written in terms of the positions of the walls in the transverse space by using $X_a \sim \varphi_a/\Lambda^2$. We can immediately cross check our result with the two wall calculation in [6]. The result (2.7) can naturally be written as a sum of a part which involves triples, (a, b, c) all distinct, and pairs, $a = c \neq b$. For a pair $a \neq b$, we have a contribution

$$\frac{g^2 m^2}{8\pi^2} \frac{\varphi_{ab}^2}{m^2 + 4\varphi_{ab}^2} , \tag{3.1}$$

which agrees with the two wall result of [6].

The result (2.7) can also be written in terms of a (real) superpotential W as

$$V(\varphi_a) = \sum_{a=1}^{n} \left(\frac{\partial W}{\partial \varphi_a}\right)^2 , \tag{3.2}$$

where

$$W_{\text{2-loop}}(\varphi_a) = \frac{2gm}{\pi} \sum_{a=1}^{n} \sum_{b=1}^{n} \left(\sqrt{4\varphi_{ab}^2 + m^2} - m\right) = \frac{4gm}{\pi} \sum_{a=1}^{n} \sum_{b=1}^{n} m_{ab}^{(-)} . \tag{3.3}$$

which makes it clear that there can be a supersymmetric bound state when all the walls are coincident but no where else.

In the limit of small separation, $|\varphi_{ab}| \ll m$, we have

$$V_{\text{2-loop}} \longrightarrow \frac{g^2 n}{32\pi^2} \sum_{a=1}^{n} \sum_{b=1}^{n} \varphi_{ab}^2 , \tag{3.4}$$

which gives the mass scale $g^2 \sim \Lambda/N$ for fluctuations around the critical point. Notice that this is small, by a factor of $1/N$ than the string scale Λ, and so should be a robust prediction from the truncated theory. In the other limit, of large separations, $|\varphi_{ab}| \gg m$, we have

$$V_{\text{2-loop}} \longrightarrow \frac{g^2 m^2}{32\pi^2} \cdot \frac{n^3 - n}{6}\,. \tag{3.5}$$

There are two remarkable things about this result: that it is a constant at all and that the result has the correct n dependence to match (1.6). This is very unexpected since it means that the relative weighting between the contributions from triples of walls and pairs of walls is exactly right. This is puzzling because it is clear that the power law behaviour of the potential for large separations cannot be physical since there is a mass gap in the 4d theory. Hence, the large distance potential must be of a Yukawa type. It seems plausible that for large separations stringy effects could change the power law to exponential behaviour; however, why should this not also effect the constant and destroy the n dependence? Note that (3.5) also has the correct N and Λ dependence once one uses the substitutions $m = g^2 N$ and $g^2 \sim \Lambda/N$.

In summary: we have calculated the domain wall potential at order $1/N$ from thinking of them as D-branes for the confining string. We expect on general grounds that the result can only really be trusted at sub-stringy distances, $X \ll \Lambda^{-1}$, and, in particular, we found a result that was consistent with the existence of a supersymmetric bound state of n walls. What is surprising is that we found a result that seems to be consistent also at very large distances: why?

Acknowledgments

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